# Technical Comments

## Comment on "Application of Biot's Variational Method to Convective Heating of a Slab"

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IN a recent note, an application of the variational principle due to Biot for heat conduction was applied to the problem of convective heating of a slab.[1] Although the results obtained in Ref. 1 are in reasonable agreement with the known exact solution, it is felt that the variational principle should be applied as outlined in an earlier paper.[2]

In particular, the application of the boundary condition should be enforced through the heat flux field $\dot{\mathbf{H}}$ and not through Fourier's heat flux relation. The use of the heat flux field to derive an over-all energy relation and the use of one of the variational equations was outlined in detail in Ref. 2. The problems discussed included those with nonlinear boundary conditions and the case of convective heating. In a later paper Richardson also has treated a class of problems with nonlinear boundary conditions, together with an application of his results.[3] Additional remarks on the use of the variational principle may be found in Refs. 4 and 5.

The purpose of this note is to indicate the application of the variational principle to a class of one-dimensional nonlinear problems in which the heat flux on the surface of a semi-infinite solid is one of the form

$$h(\theta_0^m - q_1^m) \tag{1}$$

where $h$ is a heat-transfer coefficient, $\theta_0$ is a reference temperature, $q_1$ is the surface temperature, and $m$ is a constant. It will be assumed that $h$ and $\theta_0$ are constant; an example of a case where $\theta_0$ is a function of time for $m = 1$ is given in Ref. 6. In Eq. (1), if $m = 4$, the heat flux is of the blackbody radiation type whereas, if $m = 1$, the heat flux is of the convective type. The former case was treated in Ref. 2 whereas the latter case was treated by Chu. We will obtain here the short- and long-time solutions for the surface temperature for arbitrary values of $m$.

In applying the variational method we will follow closely the procedure of Refs. 1 and 2. The assumed temperature distribution is parabolic

$$\theta = q_1[1 - (x/q_2)]^2 \tag{2}$$

where $q_1$ is the surface temperature, $x$ is the distance measured into the semi-infinite solid, and $q_2$ is the penetration depth. The variational equation

$$(\partial V/\partial q_2) + (\partial D/\partial \dot{q}_2) = Q_2 \tag{3}$$

gives

$$(\dot{\psi}\eta^2/42) + (13\psi\eta\dot{\eta}/315) = (7\psi/30) \tag{4}$$

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where

$$\psi = q_1/\theta_0 \qquad \eta = (q_2 h\theta_0^{m-1}/k)$$
$$\tau = (h\theta_0^{m-1}/k)^2\alpha t$$

and dots indicate differentiation with respect to $\tau$. All other symbols are as defined in Refs. 1 and 2. The flux condition on the surface is

$$\dot{\mathbf{H}}_n = h\theta_0^m(1 - \psi^m) \tag{5}$$

Therefore, the second equation to be used in conjunction with the variational equation (4) is the condition of over-all energy balance given by Eq. (5). Evaluation of the heat flux field gives

$$\dot{\psi}\eta + \dot{\eta}\psi = 3(1 - \psi^m) \tag{6}$$

Equations (6) and (4) are the governing equations for the time histories of the surface temperature $q_1$ and the penetration depth $q_2$.

Evaluation of the short- and long-time solutions gives

$$\tau \to 0 \qquad \eta = 2.68\,\tau^{1/2} \tag{7}$$
$$\psi = 1.12\,\tau^{1/2} \tag{8}$$
$$\tau \to \infty \qquad \eta = 3.36\,\tau^{1/2} \tag{9}$$
$$\psi = 1 - (0.56/m\tau^{1/2}) \tag{10}$$

Equations (7) and (8) agree with the previous results of Ref. 2 in which the surface temperature and penetration depth variation are given in Figs. 10 and 11.

For $m = 1$, the foregoing results can be compared with the known exact solution[1]

$$\psi = 1 - e^{\tau}\,\mathrm{erfc}\,\tau^{1/2} \tag{11}$$

The exact solutions for small and large values of $\tau$ are

$$\psi = 1.13\,\tau^{1/2} \qquad \tau \to 0 \tag{12}$$
$$\psi = 1 - 0.564/\tau^{1/2} \qquad \tau \to \infty \tag{13}$$

Comparison of Eqs. (12) and (13) with (8) and (10) indicates good agreement of the approximate variational method with the exact solution. The corresponding problem for a finite slab can be formulated easily as indicated in Ref. 2.

### References

[1] Chu, H. N., "Application of Biot's variational method to convective heating of a slab," J. Spacecraft Rockets **1**, 686–688 (1964).

[2] Lardner, T. J., "Biot's variational principle in heat conduction," AIAA J. **1**, 196–206 (1963).

[3] Richardson, P. D., "Unsteady one-dimensional heat conduction with a non-linear boundary condition," Trans. ASME, J. Heat Transfer **86**, 298–299 (1964).

[4] Goodman, T. R., "Application of integral methods to transient non-linear heat transfer," *Advances in Heat Transfer* (Academic Press, New York, 1964), Vol. 1, pp. 51–122.

[5] Lardner, T. J., "Approximate heat conduction solutions for non-planar geometries," Trans. ASME, J. Heat Transfer (to be published).

[6] Lardner, T. J. and Pohle, F. V., "Application of Biot's variational principle in heat conduction," PIBAL Rept. 587, Armed Services Technical Information Agency AD 259–696, Polytechnic Institute of Brooklyn, Brooklyn, N. Y. (1961).